FORMATION OF RIGID ZONES IN VISCOPLASTIC MEDIUM

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In this article conditions are obtained in a finite form on the boundary of rigid zones in a viscoplastic medium. A similar conditon was given in [1] but in an integral form. Moreover, a stationary flow is considered of a Bingham–Shvedov medium in a two-dimensional conduit [2]. Small orthogonal harmonic perturbations are imposed on the lower motionless conduit surface, the upper unperturbed plane moving with constant velocity and making an arbitrary angle with the perturbations. It is assumed that there is adhesion of the medium at the boundaries. The problem is solved by using the small-parameter approach. A critical condition is found by using two approximations so that rigid zones are first formed in some sections. Singularities appear in the solution if the flow is parallel to perturbations. These cases are analyzed separately. An attempt to find a criterion for starting the formation of a rigid zone in a viscoplastic medium was previously made in [3].

1. Condition on the Boundary of Rigid Zones

THEOREM. For the surface $\mathbf{v} = 0$ to be the boundary of a rigid zone for a flow of viscoplastic material ($\mathbf{v}$ is the velocity of the particle of the medium) it is necessary and sufficient that on this surface the condition

$$\partial |\mathrm{v}| / \partial n = 0 \tag{1.1}$$

be satisfied, where n is the normal coordinate from the boundary Σ of the rigid zone:

The necessary condition of the theorem can be proved by using the relations

$$\mathrm{v}|_\Sigma = 0, \quad \varepsilon_{ij}|_\Sigma = {}^1/_2\,(v_{i,j} + v_{j,i})_\Sigma = 0 \tag{1.2}$$

Sufficiency. Let the following assumptions of the theorem be satisfied on Σ: $\mathbf{v}|_\Sigma = 0$, $\partial |\mathbf{v}|/\partial n|_\Sigma = 0$. It is required to prove that $\varepsilon_{ij}|_\Sigma = 0$. The functions $\mathbf{v}$ and $\partial \mathbf{v}/\partial n$ are expanded into series in the neighborhood of the surface Σ,

$$\mathrm{v} = \partial \mathrm{v} / \partial n|_\Sigma n + \ldots, \quad \partial \mathrm{v} / \partial n = \partial \mathrm{v} / \partial n|_\Sigma + \ldots \tag{1.3}$$

$$\mathrm{v} \cdot \partial \mathrm{v} / \partial n = (\partial \mathrm{v} / \partial n|_\Sigma)^2 n + \ldots \tag{1.4}$$

For the velocity modulus these expressions can be written as

$$|\mathrm{v}| = \partial |\mathrm{v}| / \partial n|_\Sigma n + \ldots, \quad \partial |\mathrm{v}| / \partial n = \partial |\mathrm{v}| / \partial n|_\Sigma + \ldots \tag{1.5}$$

$$|\mathrm{v}|\, \partial |\mathrm{v}| / \partial n = (\partial |\mathrm{v}| / \partial n|_\Sigma)^2 n + \ldots \tag{1.6}$$

Since the left-hand sides of the expressions (1.4) and (1.6) are equal, so must also be the right-hand sides, that is,

$$(\partial |\mathrm{v}| / \partial n|_\Sigma)^2 n + \ldots = (\partial \mathrm{v} / \partial n|_\Sigma)^2 n + \ldots \tag{1.7}$$

Hence, it follows that $\partial \mathbf{v}/\partial n|_\Sigma = 0$ simultaneously with $\mathbf{v} = 0$, which indicates that $\varepsilon_{ij}|_\Sigma = 0$, which is what was required to be proved.

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2. Steady Flow of Viscoplastic Medium in a Two-Dimensional Conduit

The bottom plane of the channel ($y=0$) is not in motion and a small orthogonal perturbation

$$y = h(\delta_1 \cos m_1 x + \delta_2 \cos m_2 z) \tag{2.1}$$

is applied to it, where h is the channel width, $\delta_1, \delta_2 \ll 1$. The top plane ($y=h$) moves with constant velocity v_* parallel to the xz plane and making an angle α with the z axis.

The relation between the components σ_{ij} of the stress tensor and ε_{ij} of the strain-rates tensor in the case of the Mises plasticity conditions is given by [4]

$$\sigma_{ij} = (\sqrt{2}\,k / \sqrt{\varepsilon_{ql}\varepsilon_{ql}} + 2\eta)\varepsilon_{ij} + P\delta_{ij}, \quad \sigma_{qq} = 3P \tag{2.2}$$

where k is the flow limit, and η is the viscosity coefficient.

The components of the tensor of the strain rates are related to the flow velocity by means of

$$\varepsilon_{ij} = {}^1/_2 (v_{i,j} + v_{j,i}) \tag{2.3}$$

Only dimensionless quantities will be used from now on:

$$x = x'h,\ y = y'h,\ z = z'h,\ m_{1(2)} = m_{1(2)}'h^{-1},\ v_i = v_i'v_*,\ \sigma_{ij} = \sigma_{ij}'k,\ \varepsilon_{ij} = \varepsilon_{ij}'hv_*^{-1}$$

The equilibrium equation and the incompressibility condition for the medium are given by

$$\sigma_{,j}^{ij} = 0, \quad \operatorname{div} \mathbf{v} = 0 \tag{2.4}$$

The boundary conditions for our problem are

$$\begin{aligned} &v_x = v_y = v_z = 0 \text{ for } y = \delta_1 \cos m_1 x + \delta_2 \cos m_2 z \\ &v_x = \sin\alpha,\ v_y = 0,\ v_z = \cos\alpha \text{ for } y = 1 \end{aligned} \tag{2.5}$$

The velocity of the medium particles and the mean pressure are sought in the form

$$f(x, y, z; \delta_1, \delta_2) = f_{(y)}^{(0)} + \delta_1 f'(x, y) + \delta_2 f''(y, z) + \ldots \tag{2.6}$$

Using the boundary conditions (2.5) the zeroth approximation is the solution of the Couette problem in a two-dimensional conduit:

$$v_x^{(0)} = y\sin\alpha,\ v_y^{(0)} = 0,\ v_z^{(0)} = y\cos\alpha,\ P^{(0)} = 0 \tag{2.7}$$

Substituting the expansion (2.6) successively into (2.3), (2.2), and (2.4) and using (2.7) one obtains the following system of equations:

$$\begin{aligned} &L(2v'_{x,xx} + v'_{x,yy} + v'_{y,xy}) - \sin^2\alpha(v'_{x,yy} + v'_{y,xy}) - \sin\alpha\cos\alpha v'_{z,yy} + P'_{,x} = 0 \\ &L(v'_{y,xx} + 2v'_{y,yy} + v'_{x,xy}) - \sin^2\alpha(v'_{y,xx} + v'_{x,xy}) - \sin\alpha\cos\alpha v'_{z,xy} + P'_{,y} = 0 \\ &L(v'_{z,xx} + v'_{z,yy}) - \cos^2\alpha v'_{z,yy} - \sin\alpha\cos\alpha(v'_{x,yy} + v'_{y,xy}) = 0 \\ &v'_{x,x} + v'_{y,y} = 0 \end{aligned} \tag{2.8}$$

$$\begin{aligned} &L(v''_{x,yy} + v''_{x,zz}) - \sin^2\alpha v''_{x,yy} - \sin\alpha\cos\alpha(v''_{y,yz} + v''_{z,yy}) = 0 \\ &L(2v''_{y,yy} + v''_{y,zz} + v''_{z,yz}) - \cos^2\alpha(v''_{y,zz} + v''_{z,yz}) - \sin\alpha\cos\alpha v''_{x,yz} + P''_{,y} = 0 \\ &L(v''_{z,yy} + 2v''_{z,zz} + v''_{y,yz}) - \cos^2\alpha(v''_{z,yy} + v''_{y,yz}) - \sin\alpha\cos\alpha v''_{x,yy} + P''_{,z} = 0 \\ &v''_{y,y} + v''_{z,z} = 0 \\ &L = 1 + a^{-2},\ a^2 = kh/\eta v_* \end{aligned} \tag{2.9}$$

The boundary conditions for the systems (2.8) and (2.9) are obtained by expanding the conditions (2.5) into a series of powers of δ_1, δ_2:

$$\begin{aligned} &v_x' = -\sin\alpha\cos m_1 x,\ v_y' = 0,\ v_z' = -\cos\alpha\cos m_1 x \text{ for } y = 0 \\ &v_x' = v_y' = v_z' = 0 \text{ for } y = 1 \end{aligned} \tag{2.10}$$

$$\begin{aligned} &v_x'' = -\sin\alpha\cos m_2 z,\ v_y'' = 0,\ v_z'' = -\cos\alpha\cos m_2 z \text{ for } y = 0 \\ &v_x'' = v_y'' = v_z'' = 0 \text{ for } y = 1 \end{aligned} \tag{2.11}$$

To solve the system of equations (2.8) with the boundary conditions (1.10) the unknown functions are sought in the form

$$v_x' = A_1(y) \cos m_1 x, \quad v_y' = B_1(y) \sin m_1 x$$
$$v_z' = C_1(y) \cos m_1 x, \quad P' = D_1(y) \sin m_1 x \tag{2.12}$$

We now insert (2.12) into (2.8) and (2.10). After some transformations one obtains

$$A_1 = m_1^{-1} p B_1, \quad [m_1^{-1}(L - \sin^2\alpha) p^3 - m_1(L + \sin^2\alpha)p] B_1 -$$
$$- \sin\alpha \cos\alpha \, p^2 C_1 + m_1 D_1 = 0, [(L + \sin^2\alpha) p^2 - m_1^2 (L - \tag{2.13}$$
$$- \sin^2\alpha)] B_1 + m_1 \sin\alpha \cos\alpha \, p C_1 + p D_1 = 0, \quad \sin\alpha \cos\alpha \, (m_1^{-1} p^3 + m_1 p) B_1 + [m_1^2 L - (L - \cos^2\alpha) p] C_1 = 0$$

where the symbolic notation $p = d/dy$ has been introduced.

The boundary conditions are

$$A_1 = B_1 = C_1 = 0 \text{ for } y = 1$$
$$A_1 = -\sin\alpha, \quad B_1 = 0, \quad C_1 = -\cos\alpha \text{ for } y = 0 \tag{2.14}$$

The characteristic equation of the system (2.13) is

$$(t - 1)\{t^2 - [2 + a^2(1 + 3\sin^2\alpha)] t + 1 + a^2(1 - \sin^2\alpha)\} = 0, \quad t = \lambda^2 m_1^{-2} \tag{2.15}$$

Its roots are equal to

$$t_1 = 1, \quad t_{2,3} = {}^1/_2 \{2 + a^2(1 + 3\sin^2\alpha) \pm a [16\sin^2\alpha + a^2(1 + 3\sin^2\alpha)^2]^{1/2}\} \tag{2.16}$$

Bearing in mind that there are no multiple roots and that $\lambda_1 = -\lambda_4$, $\lambda_2 = -\lambda_5$, $\lambda_3 = -\lambda_6$, the solution of the system (2.13) is sought in the form

$$B_1(y) = \sum_1^3 [c_i \exp(\lambda_i y) + c_{i+3} \exp(-\lambda_i y)]$$
$$C_1(y) = \sum_1^3 [c_{2i} \exp(\lambda_i y) + c_{2i+3} \exp(-\lambda_i y)] \tag{2.17}$$
$$D_1(y) = \sum_1^3 [c_{3i} \exp(\lambda_i y) + c_{3i+3} \exp(-\lambda_i y)]$$

By inserting (2.17) in (2.13) one finds

$$c_{2q} = a_q c_q, \quad c_{3q} = b_q c_q$$
$$a_q = -\lambda_q(\lambda_q^2 + m_1^2) \sin\alpha \cos\alpha \, m_1^{-1} [\lambda_q^2 (L - \cos^2\alpha) - m_1^2 L]^{-1} \tag{.218}$$
$$b_q = \frac{\lambda_q L (\lambda_q^2 - m_1^2)}{m_1^2 \cos^2\alpha} \, \frac{\lambda_q^2 (1 - L) + m_1^2 (L + \sin^2\alpha)}{\lambda_q (L - \cos^2\alpha) - m_1^2 L}$$

The solution can then be written in the form

$$A_1(y) = m_1^{-1} \sum_1^3 [c_i \lambda_i \exp(\lambda_i y) - c_{i+3} \lambda_i \exp(-\lambda_i y)]$$
$$B_1(y) = \sum_1^3 [c_i \exp(\lambda_i y) + c_{i+3} \exp(-\lambda_i y)] \tag{2.19}$$
$$C_1(y) = \sum_1^3 [c_i a_i \exp(\lambda_i y) - c_{i+3} a_i \exp(-\lambda_i y)]$$
$$D_1(y) = \sum_1^3 [c_i b_i \exp(\lambda_i y) - c_{i+3} b_i \exp(-\lambda_i y)]$$

The integration constants c_i can be found from the boundary conditions (2.14),

$$\sum_1^6 c_i = 0, \quad m_1^{-1} \sum_1^3 (c_i \lambda_i - c_{i+3} \lambda_i) = -\sin\alpha$$
$$\sum_1^3 (c_i a_i - c_{i+3} a_i) = -\cos\alpha \tag{2.20}$$

$$\sum_1^3 [c_i \exp(\lambda_i) + c_{i+3} \exp(-\lambda_i)] = 0, \qquad \sum_1^3 [c_i \lambda_i \exp(\lambda_i) - c_{i+3} \lambda_i \exp(-\lambda_i)] = 0$$

$$\sum_1^3 [c_i a_i \exp(\lambda_i) - c_{i+3} a_i \exp(-\lambda_i)] = 0$$

The solution of the system (2.20) for c_1 is given by

$$A_1 = m_1^{-1} \sum_1^3 \lambda_q G_q, \quad B_1 = \sum_1^3 F_q, \quad C_1 = \sum_1^3 a_q G_q, \quad D_1 = \sum_1^3 b_q G_q \tag{2.21}$$

$$F_q = b_q^1 \operatorname{sh}(\lambda_q y) + b_q^2 \operatorname{ch}(\lambda_q y) + b_q^3 \operatorname{sh}[\lambda_q (y-1)] + b_q^4 \operatorname{ch}[\lambda_q (y-1)], \quad G_q = \lambda_q^{-1} p F_q$$

$$\begin{aligned} b_q^1 &= 2\Delta_1^{-1} (M_{12}^{qq+3} \sin\alpha - M_{13}^{qq+3} \cos\alpha) \\ b_q^2 &= 2\Delta_1^{-1} (a_q \sin\alpha - \lambda_q m_1^{-1} \cos\alpha) M_{23}^{qq+3} \\ b_q^3 &= 2\Delta_1^{-1} (M_{21}^{qq+3} \sin\alpha - M_{34}^{qq+3} \cos\alpha) \\ b_k^4 &= 2\Delta_1^{-1} [(\lambda_q M_{25}^{qq+3} - a_q M_{26}^{qq+3}) \sin\alpha - (\lambda_q M_{35}^{qq+3} - a_q M_{36}^{qq+3}) \cos\alpha] \end{aligned} \tag{2.22}$$

where (there is no summation over q) Δ_1 is the determinant of the system (2.20), M_{mn}^{qq+3} are the minors of the determinant Δ_1, the superscripts showing the deleted columns and the subscripts the deleted rows.

If the systems (2.8) and (2.9) and the boundary conditions (2.10) and (2.11) are compared it can be seen that the solution of the boundary-value problem (2.9) and (2.11) can be obtained from the solution of the problem (2.8) and (2.10) simply by replacing in the latter m_1, λ_q, $\tan\alpha$, x, z by m_2, μ_q, $\cot\alpha$, z, x, respectively; in another notation by adding an extra prime,

$$v_x'' = C_2(y) \cos m_2 z, \quad v_y'' = B_2(y) \sin m_2 z, \quad v_z'' = A_2(y) \cos m_2 z, \qquad P'' = D_2(y) \sin m_2 z \tag{2.23}$$

The functions $A_2(y)$, $B_2(y)$, $C_2(y)$, $D_2(y)$ are found from (2.21), namely,

$$A_2 = m_1^{-1} \sum_1^3 \mu_q G_q', \quad B_2 = \sum_1^3 F_q', \quad C_2 = \sum_1^3 a_q G_q', \quad D_2 = \sum_1^3 b_q' G_q' \tag{2.24}$$

$$F_q' = b_q^{1\prime} \operatorname{sh}(\mu_q y) + b_q^{2\prime} \operatorname{ch}(\mu_q y) + b_q^{3\prime} \operatorname{sh}[\mu_q (y-1)] + b_q^{4\prime} \operatorname{ch}[\mu_q (y-1)], \; G_q' = \mu_q^{-1} p F_q'$$

Similarly as in [3] the formation of rigid zones in sections on the perturbed plane is possible for some ratio of the parameters which characterize the flow of a viscoplastic medium. It can be assumed that rigid regions arise at the vertices of the sections of the lower surface ($y = -\delta_{1*} - \delta_{2*}$, $\cos m_1 x = \cos m_2 z = -1$).

By formulating the condition (1.1) for these points with an accuracy up to the small quantities of the first order a critical relation is obtained between the parameters v_*, h, k, η, α, m_{1-2}, δ_{1*}, δ_{2*},

$$1 - [A_{1,y}(0) \sin\alpha + C_{1,y}(0) \cos\alpha]\, \delta_{1*} - [C_{2,y}(0) \sin\alpha + A_{2,y}(0) \cos\alpha]\, \delta_{2*} = 0 \tag{2.25}$$

If the surface perturbation for the same flows exceeds the critical one, $\delta_1 + \delta_2 \geq \delta_{1*} + \delta_{2*}$, then rigid zones are formed in the sections, but if $\delta_1 + \delta_2 < \delta_{1*} + \delta_{2*}$, no rigid zones are found in the conduit.

For $\alpha = \pi n/2$ (n = 1, 2, ...) the obtained solution does not satisfy all the conditions of the problem, since the characteristic equation (2.15) possesses multiple or complex roots, and the solution can be much simplified.

Lengthwise Perturbations.

$$y = \delta_1 \cos m_1 x, \quad \delta_2 = 0, \quad \cos\alpha = 1$$

In this case only the velocity v_z does not vanish.
The zeroth approximation is

$$v_z^{(0)} = y \tag{2.26}$$

For the first approximation v_z' only one equation remains of the system (2.8):

$$(1 + a^2) v_{z,xx}' + v_{z,yy}' = 0 \tag{2.27}$$

From (2.10) the boundary conditions are found for (2.27) which are of the form

$$v_z'(x, 1) = 0, \quad v_z'(x, 0) = -\cos m_1 x \tag{2.28}$$

The velocity function is sought in the form

$$v_z' = C_1(y) \cos m_1 x \tag{2.29}$$

By inserting (2.29) into (2.27) and (2.28) we arrive at the equation

$$p^2 C_1 - m_1^2 (1 + a^2) C_1 = 0 \tag{2.30}$$

with the boundary conditions

$$C_1(1) = 0, \quad C_1(0) = -1 \tag{2.31}$$

Solving (2.30) and (2.31) and using (2.26) one obtains

$$v_z' = y + \delta_1 [\mathrm{sh}(\lambda)]^{-1} \mathrm{sh}[\lambda(y-1)] \cos m_1 x + o(\delta_1^2), \quad \lambda = m_1 (1 + a^2)^{1/2} \tag{2.32}$$

The criterion for the formation of stagnant zones at the vertices of the sections ($y = -\delta_{1*}$, $\cos m_1 x = -1$) is obtained from the condition (1.1) as

$$\delta_1 \geqslant \delta_{1*}, \quad \delta_{1*} = [\mathrm{th}(\lambda)] / \lambda \tag{2.33}$$

Transverse Perturbations.

$$y = \delta_2 \cos m_2 z, \quad \delta_1 = 0, \quad \cos \alpha = 1$$

In this case $v_x = 0$, $(v_z, v_y) \neq 0$.

The zeroth approximation is

$$v_y^{(0)} = 0, \quad v_z^{(0)} = y \tag{2.34}$$

From (2.9) one obtains for the first approximation the system

$$\begin{gathered} 2(1 + a^2) v''_{y,yy} + v''_{y,zz} + v''_{z,yz} + P''_{,y} = 0 \\ 2(1 + a^2) v''_{z,zz} + v''_{z,yy} + v''_{y,yz} + P''_{,z} = 0, \quad v''_{y,y} + v''_{z,z} = 0 \end{gathered} \tag{2.35}$$

The boundary conditions obtained from (1.11) are

$$v_y'' = v_z'' = 0 \text{ for } y = 1, \quad v_y'' = 0, \quad v_z'' = -\cos m_2 z \text{ for } y = 0 \tag{2.36}$$

The solution is sought in the form

$$P'' = D_2(y) \sin m_2 z, \quad v_y'' = B_2(y) \sin m_2 z, \quad v_z' = A_2(y) \cos m_2 z \tag{2.37}$$

If (2.37) is inserted into (2.35) and (2.36), then after some transformation, the problem (2.35) and (2.36) is reduced to the following system:

$$\begin{gathered} A_2 = m_2^{-1} p B_2, \quad [(1 + 2a^2) p^2 - m_2^2] B_2 + p D_2 = 0 \\ [m_2^{-2} p^3 - (1 + 2a^2) p] B_2 + D_2 = 0 \end{gathered} \tag{2.38}$$

$$A_2(1) = B_2(1) = B_2(0) = 0, \quad A_2(0) = -1 \tag{2.39}$$

The solution of (2.38) and (2.39) is given by

$$\begin{gathered} A_2 = m_2^{-1} \sum_1^2 \mu_q G_q', \quad B_2 = \sum_1^2 F_q', \quad D_2 = \sum_1^2 b_q' G_q' \\ F_q' = b_q^{1'} \mathrm{sh}(\mu_q y) + b_q^{2'} \mathrm{sh}[\mu_q (y-1)] + b_q^{3'} \mathrm{ch}[\mu_q (y-1)] \\ G_q' = \mu_q^{-1} p F_q', \quad b_q' = \mu_q [\mu_q^2 m_2^{-2} - (1 + 2a^2)] \\ \mu_q^2 = m_2^2 t_q, \quad t_{1,2} = 1 + 2a^2 \pm 2a (1 + a^2)^{1/2} \end{gathered} \tag{2.40}$$

where $b_q^{n\prime}$ represent the expressions

$$b_q^{1'} = -2\Delta_2^{-1} M_{12}^{qq+2}, \quad b_q^{2'} = 2\Delta_2^{-1} M_{23}^{qq+2}, \quad b_q^{3'} = 2\Delta_2^{-1} \mu_q M_{24}^{qq+2}$$

(there is no summation over q)

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \mu_1 & \mu_2 & -\mu_1 & -\mu_2 \\ M_1 & M_2 & M_1^{-1} & M_2^{-1} \\ \mu_1 M_1 & \mu_2 M_2 & -\mu_1 M_1^{-1} & -\mu_2 M_2^{-1} \end{vmatrix}, \quad M_i = \exp(\mu_i)$$

where M_{mn}^{qq+2} denote the minors of the determinant Δ_2.

In this case the speed is

$$|\mathbf{V}| = y + \delta_2 A_2(y)\cos m_2 z + o(\delta_2^2) \tag{2.14}$$

The criterion for the formation of stagnant zones at the points of the surface ($y = -\delta_{2*}$, $\cos m_2 z = -1$) is as follows:

$$\delta_2 \geqslant \delta_{2*}, \quad \delta_{2*} = \left\{\sum_1^2 [\mu_q^2 (b_q^{3\prime} \operatorname{ch} \mu_q - b_q^{2\prime} \operatorname{sh} \mu_q)]\right\}^{-1} \tag{2.42}$$

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